A theory of evolving natural constants embracing Einstein's theory of general relativity and Dirac's large number hypothesis

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Abstract

Taking a hint from Dirac's large number hypothesis, we note the existence of cosmic combined conservation laws that work to cosmologically long time. We thus modify or generalize Einstein's theory of general relativity with fixed gravitation constant G to a theory for varying G, which can be applied to cosmology without inconsistency, where a tensor arising from the variation of G takes the place of the cosmological constant term. We then develop on this basis a systematic theory of evolving natural constants $m_e, m_p, e, \hbar, k_B$ by finding out their cosmic combined counterparts involving factors of appropriate powers of G that remain truly constant to cosmologically long time. As G varies so little in recent centuries, so we take these natural constants to be constant.

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I. GENERAL RELATIVITY AND LARGE NUMBER HYPOTHESIS

In Einstein's theory of general relativity as in Newton's theory of gravitation, the strength of the gravitational interaction is described by a fixed dimensional constant, namely the Newtonian gravitation constant $G_N \doteq 6.7 \times 10^{-11} \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. Together with the velocity of light in vacuo $c \doteq 3 \times 10^8 \mathrm{m/s}$ which serves to convert time into length by putting $c = 1$, there occur also in Einstein's theory $G_N/c^2 \doteq 7.4 \times 10^{-28} \mathrm{mkg}^{-1}$ and $G_N/c^4 \doteq 8.2 \times 10^{-45} \mathrm{mJ}^{-1}$ which respectively serve to convert mass and energy into length by putting also $G_N = 1$. Einstein's theory has been applied to cosmology; in addition to the dimensional constants G_N another cosmological constant with dimension of length to the power minus two, was once specially introduced by Einstein in his attempt to construct a static model for the universe. This soon has lost his favor as observations of the red shifts of extra-galactic nebulae show defintely that the universe is expanding, but as an additional parameter, cosmological constant is still used today in trying to fit the observational data with the non-static homogeneous cosmological model using the Roberston-Walker metric. Rough estimates have thus been obtained for the age t of universe at present to be $\approx 10^9$ years or $\approx 10^{10}$ years more recently.

On comparing the ratio $e^2/(Gm_pm_e)$ of the electrostatic to the gravitational force between the proton and the electron in an hydrogen atom, which is a large dimensionless number of the order of 10^{39} , with the ratio $t/(e^2/m_ec^3)$ of the age t of universe at present to the time needed for light to travel a distance of the classical radius of the electron, which is also a large dimensionless number of the order of 10^{39} or 10^{40}, one proposes [1] that a large number equation

$$e^2/(Gm_pm_e) \approx t/(e^2/m_ec^3) \qquad (1.1)$$

should hold. Further from Hubble's estimate $\rho = (1.3 \text{ to } 1.6) \times 10^{-30}$ g/cm^3 for the density of matter due to the extra-galactic nebulae averaged over cosmic space and a factor thousand or hundred times to include dust or invisible matter, one can estimate the ratio M/m_p of the total mass M of matter in the universe at present of radius R to the proton mass, which is roughly of the order 10^{78} or 10^{80}, one proposes [1] that another large number equation

$$M/m_p \approx [t/(e^2/m_ec^3)]^2 \qquad (1.2)$$

should also hold. By a large number equation we mean wherein some unknown factors of small numbers are understood to be there but not written out. Eddington takes seriously the relation

$$M/m_H = N \approx [e^2/(Gm_p m_e)]^2 \approx 10^{78} \quad (1.3)$$

and advocates in his book "Fundamental Theory" that in the uranoid (model of universe with N hydrogen atoms) when one studies one particular hydrogen atom the other $N-1$ hydrogen atoms must also be considered as representing the rest of the universe.

Based on these empirical large number equations (1.1) and (1.2) Dirac proposes the large number hypothesis [1] that during the evolution of the universe the strength G of gravitational interaction gradually weakens as time t goes on according to $G \propto t^{-1}$ while mass is continuously being created and so increasing with time according to $M \propto t^2$. He reached at this hypothesis by adding an additional assumption that the atomic constants remain unchanged and do not evolve. One can, however, arrive at Dirac's large number hypothesis by the following evolving atomic constants, say, both m_p and $m_e \propto t^2$ while $e^2 \propto t^3$, leaving c alone not evolving. This corresponds to Eddington's assumption that N remains unchanged.

In this paper we shall take the large number equations, though empirical at the present stage of knowledge, rather seriously. On dividing (1.2) by (1.1) we obtain a relation free from all atomic constants e, m_e, m_p,

$$GM \approx c^3 t \qquad (1.4)$$

which exhibits clearly the inconsistency of applying Einstein's theory of general relativity to cosmology. If one regard this result, though empirical, as true, then clearly G and M cannot both conserve in cosmologically long time, and this is in contradiction to the assumptions inherent in Einstein's theory of general relativity that G be constant and matter do conserve. Hence one needs to modify Einstein's theory for application to cosmology. But we know, for phenomena including the crucial tests that occur, from the cosmic point of view, in a small neighborhood and for short duration, Einstein's theory is a good approximation. This has been confirmed by observations and experiments and must not be spoiled. We note, according to Dirac's large number hypothesis, it is neither M nor G but the combination G^2M that conserves in cosmologically long time. We shall take such cosmic combined (c.c.)

conservation law working to cosmologically long time as a hint in our attempts to modify or rather to generalize Einstein's theory of general relativity, to be consistent with Dirac's large number hypothesis, and develop systematically a theory of evolving natural constants. In so doing we find that it is possible to generalize Dirac's large number hypothesis somewhat to $G \propto t^{-n}$ with n not necessarily equal to one. It seems that c. c. conservation laws can be more simply found and stated, but the variation of G, together with that of $g_{\mu\nu}$, as functions of space-time must be left to the solution of various peoblems.

II. THEORY WITH VARYING G

The problem of non-static homogeneous matter-dominated cosmological model is a simple case to work with mathematically. Because there is only one independent variable t, it is easier here to find out how natural constants shall evolve and how the usual conservation laws shall be combined to work in cosmologically long time. We shall not make Dirac's assumption that atomic constants do not change in the long. (In this paper by long or short we mean cosmologically long or short time and similarly by large or small we mean cosmologically large or small distance). But we do follow the spirit of Dirac's large number hypothesis and generalize it into

$$G \propto t^{-n}, (0 < n < 2), \text{ so } t \propto G^{-1/n} \quad (2.1)$$

where n can be, but not necessarily, equal to one. Then (1.3) leads to

$$G^{1+1/n} M \propto t^0 \text{ i.e. } G^{1+1/n} M = \widetilde{M} \text{ conserves.} \quad (2.2)$$

To save one's worry about the dimension of the cosmic combined (c.c.) quantity like $\widetilde{M}$, it would be most simple to introduce a constant but arbitrary dimensional conversion factor G_0 to convert mass into length by putting it to be $G_0 = 1$, so that in (2.2) and similar equations in the following we understand that the c. c. quantity (like $\widetilde{M}$) is always of the same dimension as the original quantity (like M). Thus the equation (2.2) is meant to be, when writen in full,

$$(G/G_0)^{1+1/n} M = \widetilde{M} \quad (2.2a)$$

As usual we take G_N for G_0. In order to conform to such a c. c. conservation law for $\widetilde{M}$ we modify the corresponding action integral for simple matter without internal stress which in

Einstein's theory is given, as shown in Dirac's book [2] on general theory of relativity, by

$$I_m = -\int (g_{\mu\nu}\wp^\mu\wp^\nu)^{1/2}d^4x \quad \text{(E2.3a) with constraint } \wp^\mu{}_{,\mu} = 0 \quad \text{(E2.3b)}$$

into a similar expression

$$\widetilde{I}_m = -\int (g_{\mu\nu}\widetilde{\wp}^\mu\widetilde{\wp}^\nu)^{1/2}d^4x \text{ (2.3a) with modified constraint } \widetilde{\wp}^\mu{}_{,\mu} = 0 \quad \text{(2.3b)}$$

Here with a change of notation

$$\begin{aligned} u^\mu &= \wp^\mu/(g_{\alpha\beta}\wp^\alpha\wp^\beta)^{1/2} \quad \text{(E2.4a), so } u^\mu u_\mu = 1 \\ (g_{\alpha\beta}\wp^\alpha\wp^\beta)^{1/2} &= \rho_m\surd(-g) \quad \text{so } \wp^\mu = u^\mu\rho_m\surd(-g) \quad \text{(E2.4b)} \end{aligned}$$

the constraint in Einstein's theory gives the conservation of mass and may be written as the equation of continuity

$$\int \wp^4 dx^1dx^2dx^3 = M \quad \text{(E2.4c) } (\rho_m u^\mu)_{;\mu} = 0 \text{ (E2.4d)}$$

Similarly with a change of notation with the cosmic combined quantities

$$(g_{\alpha\beta}\widetilde{\wp}^\alpha\widetilde{\wp}^\beta)^{1/2} = \widetilde{\rho}_m\surd(-g) \quad \text{(2.4a)} \quad \widetilde{\wp}^\mu = u^\mu\widetilde{\rho}_m\surd(-g) \quad \text{(2.4b)}$$

where

$$\widetilde{\rho}_m = G^{1+1/n}\rho_m \quad (2.5)$$

the modified constraint (2.3b) in the modified theory gives the conservation of the cosmic combined mass

$$\int \widetilde{\wp}^4 dx^1dx^2dx^3 = \widetilde{M} = G^{1+1/n}M \quad \text{(2.4c)(2.5c)}$$

for homogeneous $G = G(t)$ and the corresponding equation of continuity

$$(\widetilde{\rho}_m u^\mu)_{;\mu} = 0 \text{ (2.4d)}$$

In case of a mass point described with a three dimensional delta function factor in (E2.4c) and (2.4c), we obtain from (2.5c) the cosmic combined constant mass

$$\widetilde{m} = G^{1+1/n}m \quad \text{(2.5a)}$$

which holds for a proton, an electron, an atom or a molecue, contrary to Dirac's assumtion that atomic constants like m_e and m_p do not evolve. From this we see that our modification

or generalization of Einstein's theory consists of multiplying the integrand of the source action integral by the dimensionless varying factor $G^{1+1/n}$ and introducing new variables suitable for expressing the modified cosmic combined conservation law. This rule will be kept throughout for other action integrals involved in the comprehensive action principle, cf. §6, §8 and §12 later.

As to Einstein's original gravitational action integral where G^{-1} is constant

$$I_g = (16\pi)^{-1} \int G^{-1} R^\sigma_\sigma \sqrt{(-g)} d^4x \text{ where } R^\sigma_\upsilon = R_{\mu\nu} g^{\mu\nu} \quad \text{(E2.6)}$$

we multiply the integrand by the factor $G^{1+1/n}$ and add a kinetic term thus:

$$\widetilde{I}_g = (16\pi)^{-1} \int [G^{1/n} R^\sigma_\sigma - w(G^{1/(2n)})_{;\sigma}(G^{1/(2n)})^{;\sigma}] \sqrt{(-g)} d^4x \quad (2.6)$$

This is the most general expression for an action integral involving only field variables $G, g_{\mu\nu}$ without any dimensional constants that will reduce to Einstein's I_g in case G be a constant. As the numerical factor $(16\pi)^{-1}$ in Einstein's theory is determined by comparison with Newton's theory in the so called non-relativistic approximation (which is good for phenomena with velocity small compared with that of light), so the factor $(16\pi)^{-1}$ in our modified theory is determined by comparison with Einstein's theory in the non-cosmologic approximation (which is good for phenomena of duration and distance short compared with that of the universe, as will be shown in §5). The numerical constant w will be determined from the non-static homogeneous matter-dominated cosmological model in §4 by using our generalized large number hypothesis (2.1). We prefer the choice of $w = 8$ for the case $k = 0$ of the Robertson-Walker metric.

In later sections we shall see that such a modification or generalization can be carried through other source action integrals like that for Maxwell's theory of electromagnetism (§6) or that for Dirac's theory of electron (§8). In §12 we consider statistical mechanics. We thus arrive at a systematic theory of evolving natural constants passing all the crucial tests of Einstein's theory of general relativity and agreeing with the empirical large number equalities (1.1), (1.2) or (1.4) when applied to cosmology. We leave the exact numerical value n introduced in the generalized Dirac's large number hypothesis to be determined in the future by fitting accurate relevant astrophysical or cosmological observations. For the moment one may take $n = 1$, the only integer in tne domain $0 < n < 2$, if one likes, following Dirac's original large number hypothesis.

III. VARIATIONAL EQUATIONS

It is convenient (cf.(2.6) and (2.1)) to introduce the dimensionless new variable

$$\phi = G^{1/(2n)} \quad , \quad \phi^2 = G^{1/n} \quad (3.1)$$

so that (2.6) becomes

$$\widetilde{I}_g = (16\pi)^{-1} \int (\phi^2 R_{\mu\nu} - w\phi_{;\mu}\phi_{;\nu}) g^{\mu\nu} \sqrt{(-g)} d^4x \quad (3.2)$$

We obtain the variation as

$$\delta \widetilde{I}_g = (16\pi)^{-1} \int [-N^{\mu\nu}\delta g_{\mu\nu} + 2\Phi\delta\phi] \sqrt{(-g)} d^4x \qquad (3.3)$$

with

$$\begin{aligned} N^{\beta}_{\alpha} \; &= \; \phi^2 [R^{\beta}_{\alpha} - (1/2) R^{\sigma}_{\sigma} \delta^{\beta}_{\alpha}] + (\phi^2)^{;\beta}_{;\alpha} - (\phi^2)^{;\sigma}_{;\sigma} \delta^{\beta}_{\alpha} \\ &- w[\phi_{;\alpha}\phi^{;\beta} - (1/2)\phi_{;\sigma}\phi^{;\sigma}\delta^{\beta}_{\alpha}] \qquad (3.4) \end{aligned}$$

(here we use the Palatini identity to express $\delta R_{\mu\nu}$ in terms of the second covariant derivatives of $\delta g_{\alpha\beta}$ and integrate by parts twice), and

$$\Phi = R^{\sigma}_{\sigma}\phi + w\phi^{;\sigma}_{;\sigma} \qquad (3.5)$$

We note the identities (which can be shown from (3.3) by using an infinitesmal transformation of coordinates or explicitly verified from (3.4) and (3.5) by using the formula $(A^{\lambda})_{;\mu;\nu} \; - (A^{\lambda})_{;\nu;\mu} = - \; A^{\sigma} R^{\lambda}_{\ldots\sigma\mu\nu}$ with $\lambda = \nu$ summed), namely

$$N^{\beta}_{\alpha;\beta} + \Phi\phi_{;\alpha} = 0 \qquad (3.6)$$

To calculate the variation of $\widetilde{I}_m$, (2.3a), we follow closely Dirac's treatment for I_m, (E2.3a), as given in his book [2] on general theory of relativity. (Only we use the letter u instead of Dirac's v to denote the four-dimensional velocity)

$$\delta \widetilde{I}_m = -(1/2) \int \widetilde{T}^{\mu\nu} \delta g_{\mu\nu} \sqrt{(-g)} d^4x - \int u_{\mu} \delta \widetilde{\wp}^{\mu} d^4x \quad (3.7)$$

with

$$\widetilde{T}^{\mu\nu} = (g_{\alpha\beta}\widetilde{\wp}^{\alpha}\widetilde{\wp}^{\beta})^{-1/2} \widetilde{\wp}^{\mu}\widetilde{\wp}^{\nu} / \sqrt{(-g)} = \widetilde{\rho}_m u^{\mu} u^{\nu} \quad (3.8)$$

by a change of notation that

$$u^\mu = (g_{\alpha\beta}\widetilde{\wp}^\alpha\widetilde{\wp}^\beta)^{-1/2}\widetilde{\wp}^\mu \quad (3.9)$$

$$\widetilde{\rho}_m\surd(-g) = (g_{\alpha\beta}\widetilde{\wp}^\alpha\widetilde{\wp}^\beta)^{1/2} \quad (3.10)$$

The second part of $\delta\widetilde{I}_m$, after substituting

$$\delta\widetilde{\wp}^\mu = (\widetilde{\wp}^\nu b^\mu - \widetilde{\wp}^\mu b^\nu),_\nu \ \ (3.11) \quad \text{so} \quad \delta\widetilde{\wp}^\mu,_\mu = 0 \ \ (3.12)$$

in compliance with the above constraint (2.3b), where b^μ is the four-dimensional virtual displacement of the element of matter, becomes after integrating by parts

$$\begin{aligned} -\int u_\mu \delta\widetilde{\wp}^\mu d^4x &= -\int u_\mu(\widetilde{\wp}^\nu b^\mu - \widetilde{\wp}^\mu b^\nu),_\nu d^4x \\ &= \int (u_\mu,_\nu - u_\nu,_\mu)\widetilde{\wp}^\nu b^\mu d^4x \qquad (3.13) \end{aligned}$$

The vanishing of this integral for an arbitrary b^μ gives the geodesic equation of motion for the element of matter, namely

$$\widetilde{\rho}_m u^\nu(u_\mu,_\nu - u_\nu,_\mu) = \widetilde{\rho}_m u^\nu(u_{\mu;\nu} - u_{\nu;\mu}) = \widetilde{\rho}_m u^\mu u_{\nu;\mu} = 0 \ \ (3.14)$$

the other part vanishing identically because we have from (3.9) the identity $u^\nu u_\nu = 1$ so $u^\nu u_{\nu;\mu} = 0$. Here we note the identies similar to (3.6)

$$\widetilde{T}^\mu_{\nu;\mu} = (u^\mu u_\nu)_{;\mu} = (\widetilde{\rho}_m u^\mu)_{;\mu}u_\nu + \widetilde{\rho}_m u^\mu u_{\nu;\mu} = 0 \quad (3.15)$$

The identities (3.6) and (3.15) show that the system of variational equations obtained from the comprehensive action principle $\delta\widetilde{I}_{tot} = \delta\widetilde{I}_g + \delta\widetilde{I}_m = 0$, namely the gravitational field equations for $g_{\mu\nu}$ and ϕ,

$$\begin{aligned} N^\beta_\alpha &= \phi^2[R^\beta_\alpha - (1/2)R^\sigma_\sigma\delta^\beta_\alpha] + (\phi^2)^{;\beta}_{;\alpha} - (\phi^2)^{;\sigma}_{;\sigma}\delta^\beta_\alpha - w[\phi_{;\alpha}\phi^{;\beta} - (1/2)\phi_{;\sigma}\phi^{;\sigma}\delta^\beta_\alpha] \\ &= -8\pi\widetilde{T}^\beta_\alpha = -8\pi\widetilde{\rho}_m u_\alpha u^\beta \qquad (3.16) \end{aligned}$$

$$\Phi = (w\phi^{;\sigma}_{;\sigma} + R^\sigma_\sigma\phi) = 0 \quad (3.17)$$

and the equations of motion (3.14) for the elements of matter are compatiable but indeterminate. This is only natural for invariant action integrals because the system of equations must allow $g_{\mu\nu}$ to change with transformation of coordinates.

We note the following combination of (3.16) and (3.17) gives a particular simple equation,

$$N_\sigma^\sigma + \phi\Phi = (w/2-3)(\phi^2)_{;\sigma}^{;\sigma} = -8\pi\widetilde{T}_\sigma^\sigma = -8\pi\widetilde{\rho}_m \quad (3.18)$$

It is interesting to write the field equations for $g_{\mu\nu}$ in a form comparable with that in Einstein's theory involving a cosmological term. Substituting (3.4) into (3.16) and dividing both sides by ϕ^2 we note that on the right-hand-side we have from (2.5) and (3.1)

$$\widetilde{\rho}_m/\phi^2 = G\rho_m \quad (3.19) \text{ or } \widetilde{T}_\alpha^\beta/\phi^2 = GT_\alpha^\beta \quad (3.20)$$

Hence we obtain

$$R_\alpha^\beta - (1/2)R_\sigma^\sigma\delta_\alpha^\beta + \Lambda_\alpha^\beta = -8\pi GT_\alpha^\beta \quad (3.21)$$

where on the right G is not a constant but varies as t^{-n} while on the left

$$\begin{aligned}\Lambda_\alpha^\beta &= [(\phi^2)_{;\alpha}^{;\beta} - (\phi^2)_{;\sigma}^{;\sigma}\delta_\alpha^\beta]/\phi^2 \\ &\quad -w[\phi_{;\alpha}\phi^{;\beta} - (1/2)\phi_{;\sigma}\phi^{;\sigma}\delta_\alpha^\beta]/\phi^2 \quad (3.22)\end{aligned}$$

differs in its tensor character from Einstein's $\Lambda\delta_\alpha^\beta = -\lambda\delta_\alpha^\beta$ and is originated from the variation of G , remembering (3.1). In case $G = const$ in a region, Λ_α^β vanishes in the same region. (Here we use the capital Greek Λ_α^β in (3.21) to compare with the similar equation in Tolman's book [3] on relativity, thermodynamics and cosmology which we write here with G_N restored and one index lowered

$$-8\pi G_N T_\alpha^\beta = R_\alpha^\beta - (1/2)R_\sigma^\sigma\delta_\alpha^\beta + \Lambda\delta_\alpha^\beta$$

The corresponding equation in Weinberg's book [4] on gravitation and cosmology is

$$R_\alpha^\beta - (1/2)R_\sigma^\sigma\delta_\alpha^\beta - \lambda\delta_\alpha^\beta = -8\pi G_N T_\alpha^\beta$$

so $\lambda = -\Lambda$. To compare with the latter we should use $\lambda_\alpha^\beta = -\Lambda_\alpha^\beta$)

IV. MATTER DOMINATED COSMOLOGICAL MODEL

We can adopt as usual the Robertson-Walker metric because in simplifying the metric to this form only considerations on symmetry and freedom of coordinate transformation have been used, but no use is made of the field equations. With t denoting the cosmic

time and r, θ, φ the dimensionless co-moving coordinates, $ds^2 = dt^2 - R^2(t)\{dr^2/(1-kr^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\}$, $k = 1, 0,$ or -1 all the equations of motions for the element of matter are trivially satisfied by $u^4 = u_4 = 1, u^i = u_i = 0$ i.e. by being at rest in the co-moving coordinates. From the modified constraint (2.3b) we obtain, using dots for time derivatives,

$$\{\widetilde{\rho}_m R^3[(1-kr^2)^{-1/2} r^2 \sin\theta]\}^{\cdot} = 0 \text{ i.e. } \widetilde{\rho}_m = \widetilde{\rho}_m(t_0) R^3(t_0)/R^3 \quad (4.1)$$

where the constant t_0 is really arbitrary but we often choose it to be the time or the age of universe at present. Then (3.18) can be integrated to give

$$R^3(\phi^2)^{\cdot} = (3 - w/2)^{-1} 8\pi \widetilde{\rho}_m(t_0) R^3(t_0) t \quad (4.2)$$

Here the constant of integration additive to t is chosen to be zero so that t is counted since the big bang when $R = 0$. We know according to the generalized large number hypothesis (2.1) that by (3.1) $\phi^2 \propto t^{-1}$ in non-static homogeous cosmological model, hence (4.2) shows that $R \propto t$. Writing proportions as equalities

$$\begin{aligned} \phi^2 &= t_0 \phi^2(t_0)/t \quad (4.3) \\ R/R(t_0) &= t/t_0 \ (4.4) \text{ or } R = \beta_{ex} t \ \ (4.5) \end{aligned}$$

β_{ex} being the dimensionless proportional constant of the expanding universe, we see that (4.2) is satisfied and gives

$$8\pi \widetilde{\rho}_m(t_0)/\phi^2(t_0) = 8\pi G(t_0)\rho_m(t_0) = (w/2 - 3)/t_0^2 \quad (4.6)$$

Substituting (4.3) and (4.4a,b) into (3.17), (3.16), we see all the field equations are satisfied if we choose for our model with

$$w = 8(1 + k/\beta_{ex}^2) \quad (4.7)$$

For example the field equation (3.17), namely

$$R^\mu_\mu \phi + w\phi^{;\mu}_{;\mu} = 6[\ddot{R}/R + (\dot{R}^2 + k)/R^2]\phi + w[\ddot{\phi} + 3(\dot{R}/R)\dot{\phi}] = 0 \qquad (4.8)$$

and the field equations $N^1_1 = N^2_2 = N^3_3 = 0$ which all coincide into, cf.(3.16),

$$\begin{aligned} N^1_1 &= \phi^2[-(\dot{R}^2 + k)/R^2 - 2\ddot{R}/R] - (\phi^2)^{\cdot\cdot} - 2(\phi^2)^{\cdot}\dot{R}/R \\ + w\dot{\phi}^2/2 &= 0 \qquad (4.9) \end{aligned}$$

both agree to determine w as above, while the equation

$$\begin{aligned} N_4^4 &= \phi^2[-3(\dot{R}^2 + k)/R^2] - 3(\phi^2)^{\cdot}\dot{R}/R \\ -w\dot{\phi}^2/2 &= -8\pi\rho_m^C \quad (4.10) \end{aligned}$$

gives a relation similar to (4.6) but with the factor $(w/2-3)$ replaced by $(w/8+3k/\beta_{ex}^2)$. These two factors, however, agree to be equal to $(1+4k/\beta_{ex}^2)$ by (4.7). Thus (4.3) and (4.5) are indeed the solution of our non-static homogeneous cosmologic model, where (see (4.6),(4.7))

$$8\pi G(t_0)\rho_m(t_0)t_0^2 = 1 + 4k/\beta_{ex}^2 = 8\pi G\rho_m t^2, \quad (4.11)$$

holds for arbitrary t_0 or t. Together with $4\pi\rho_m R^3/3 = M$, we obtain from (4.11)

$$GM = (\beta_{ex}^3/6)(1 + 4k/\beta_{ex}^2)c^3 t \quad (4.12)$$

in agreement with the large number relation (1.4) from where we started. With (2.1) and (2.2), (4.12) becomes simply the conservation of $\widetilde{M}$. Also (4.11) shows by (2.1) that $\rho_m \propto t^{-(2-n)}$ which suggests the domain for n given in (2.1) above. We have from (3.22) and (4.3)(4.4) , or cf. (4.9) and (4.10),

$$\begin{aligned} \Lambda_1^1 &= \Lambda_2^2 = \Lambda_3^3 = (1 + k/\beta_{ex}^2)/t^2 \\ \Lambda_4^4 &= (2 - k/\beta_{ex}^2)/t^2 \qquad (4.13) \end{aligned}$$

We believe that the action integral $\widetilde{I}_g$ should be given once for all, being independent of the problems as here treated. From (4.6) we must choose $w > 6$, and this excludes by (4.7) the case of $k = -1$ for $\beta_{ex}^2 < 1$. For the same reason the case of $k = +1$ can occur only for $w \geq 16$. Our first preference is to choose $w = 8$, thereby the case of flat three dimensional space $k = 0$ is determined. The second choice $w = 16$, which corresponds to the case $k = +1$ and $\beta_{ex} = 1$, is far less probable, we think. This question may be settled by noting the difference in (4.13), i.e.

$$\begin{aligned} \Lambda_4^4/\Lambda_1^1 &= 2 \text{ for } w = 8 \quad (4.13a) \\ \Lambda_4^4/\Lambda_1^1 &= 1/2 \text{ for } w = 16 \ (4.13b) \end{aligned}$$

Of course a fresh analysis of the data of modern observation according to the formulae of the present theory would be very important. Our cosmology term varies inversely with the

square of the cosmic time. In terms of the Hubble function obtained from (4.4) (please note the different numerical coefficient from Einstein's theory)

$$H(t) = \dot{R}(t)/R(t) = 1/t \quad (4.14)$$

both the age of universe t_0 and the values of our cosmological term at present can be determined from the Hubble constant $H_0 = H(t_0)$ defined by small cosmic redshifts as discussed in §9. This seems to give a natural explaination for the order of magnitude of the cosmological constant at present.

V. NON-COSMOLOGICAL APPROXIMATION

We are intrerested in examine whether our modified theory may pass the crucial tests that support Einstein's theory of general relativity. Here we shall consider in particular that refering to the advance of perihelion of Mercury. We need only the exterior solution for $g_{\mu\nu}$ and ϕ good for, cosmologically speakng, a small region of space around the sun and a short inteval Δt around t_0 say the present epoch. The motion of the planet e.g. Mercury will then be obtained from the geodesic equations of motion (3.4). We consider at first the cosmic background, i.e. the cosmological model studied in §4. We write the dimensionless co-moving coordinate in the Robertson-Walker metric in this section as r_{R-W} and introduce a new variable r of dimension length

$$r = R(t_0) r_{R-W} \quad (5.1)$$

The metric becomes for example in the case of $k = 0$

$$ds^2 = dt^2 - [R^2(t)/R^2(t_0)](dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \quad (5.2)$$

To the non-cosmological approximation we neglect Δt against t_0 and so, by (4.4),

$$R^2(t)/R^2(t_0) = t^2/t_0^2 = 1 + 2\Delta t/t_0 \doteq 1 \quad (5.3)$$

the space-time becomes flat, the background expansion being insignificant. The non-cosmological approximation can simply be expressed as $t_0^{-1} \to 0$, in the sense that terms involving t_0 in the denominator can be neglected. To this approximation G (hence ϕ) assumes its quasistatic value, for example

$$G(t)/G(t_0) = (t/t_0)^{-n} = 1 - n\Delta t/t_0 \doteq 1 \quad (5.4)$$

This is consistent with $\Lambda_\alpha^\beta = O(t_0^{-2}) \to 0$ by (4.13) and $G\rho_m = O(t_0^{-2}) \to 0$ by (4.11), i.e. the background becomes the vacuum and cosmological terms disappear.

Now consider the sun as a local concentration of matter with density $\rho(r)$ much higher than the cosmic background.. To the non-cosmological approximation we can solve the system of gravitational field equations (3.16) and (3.17) for $g_{\mu\nu}$ and ϕ outside the sun by the quasistatic $(t/t_0 = 1 + \Delta t/t_0 = 1)$ exterior solution

$$\text{Exterior} \quad \phi(t,r) = \phi(t_0) = const \quad (5.5) \quad R_{\mu\nu} = 0 \quad (5.6)$$

with the boundary condition at spatial infinity $g_{\mu\nu}$ tending to, according to (5.3), $\eta_{\mu\nu}$, that of flat space-time. Thus we see, as far as exterior solution are concerned, our theory reduces to Einstein's theory in the non-cosmological approximation. So our theory passes the crutial test about the advance of perihelion.

VI. ELECTROMAGNETISM FOR VARYING G

In this section we shall go to find the modification for varying G of the action integral for the electromagnetic field and the additional action integral for charged matter. We start from the large number equation (1.2). According to our approach of replacing evolving law by cosmic combined conservation law, for example see (2.4)(2.5) for $\widetilde{I}_m$ for matter, which we consider it applicable also to the sun, the planet, even to a mass point, so

$$\widetilde{M} = G^{1+1/n} M, \ \widetilde{m}_p = G^{1+1/n} m_p, \ \widetilde{m}_e = G^{1+1/n} m_e \quad (6.1)$$

and the left hand side of (1.2) is a fixed constant. Assuming c not evolving, we conclude that $e^2/m_e \propto t \propto G^{-1/n}$, so the cosmic combined charge

$$\widetilde{e} = G^{1/2+1/n} e \quad (6.2)$$

conserves in the long.

In Einstein's theory with constant G, the action integral for the electromagnetic field is (we follow again closely Dirac's book on general theory of relativity)

$$I_{em} = -(16\pi)^{-1} \int F_{\mu\nu} F^{\mu\nu} \sqrt{(-g)} d^4x \ \text{ where } F_{\mu\nu} = \kappa_{\mu;\nu} - \kappa_{\nu;\mu} \ \text{(E6.3a,b)}$$

and the additional action integral for charged matter is (cf Dirac (29.4)(29.1))

$$I_q = -\int \kappa_\mu \hat{\mathrm{J}}^\mu d^4x \ \text{ or } \ = -e \int \kappa_\mu dx^\mu \ \text{ (E6.4a,b)}$$

for continuous distribution of charges or for a point charge. Recalling the rule of our modification mentioned before in connection with I_m and $\widetilde{I}_m$, i.e. mutiplying I with $G^{1+1/n}$ and keep an eye on the conservation law, we get naturally, starting from (6.2) and (E6.4b), then (E6.4a) or (E6.3b), and finally (E6.3a), the following modified action integrals

$$\widetilde{I}_q = -\widetilde{e}\int \widetilde{\kappa}_\mu dx^\mu \text{ with } \widetilde{\kappa}_\mu = G^{1/2}\kappa_\mu \quad (6.4\text{b,c})$$

$$\widetilde{I}_q = -\int \widetilde{\kappa}_\mu \widetilde{\jmath}^\mu d^4x \text{ with } \widetilde{\jmath}^\mu = G^{1/2+1/n}\jmath^\mu \text{ (6.4a,d)}$$

$$\widetilde{F}_{\mu\nu} = \widetilde{\kappa}_{\mu;\nu} - \widetilde{\kappa}_{\nu;\mu} \ = \widetilde{\kappa}_{\mu},_{v} - \widetilde{\kappa}_{\nu},_{\mu} \quad (6.3\text{b})$$

$$\widetilde{I}_{em} = -(16\pi)^{-1}\int \phi^2 \widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}\sqrt{(-g)}d^4x \quad (6.3\text{a})$$

We give the variations of (6.3a)

$$\begin{aligned}\delta\widetilde{I}_{em} \ &= \int [(-1/2)\widetilde{E}^{\mu\nu}\delta g_{\mu\nu} + (4\pi)^{-1}(\phi^2\widetilde{F}^{\mu\nu})_{;\nu}\delta\kappa_\mu \\ &+\Phi_{em}\delta\phi]\sqrt{(-g)}d^4x \quad (6.5\text{a})\end{aligned}$$

with

$$\widetilde{E}^\nu_\mu = \phi^2[-(4\pi)^{-1}\widetilde{F}_{\mu\sigma}\widetilde{F}^{\nu\sigma} + (16\pi)^{-1}\widetilde{F}_{\rho\sigma}\widetilde{F}^{\rho\sigma}\delta^\nu_\mu \quad (6.5\text{b})$$

$$\Phi_{em} = -(8\pi)^{-1}\phi\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} \quad (6.5\text{c})$$

The variation of (6.4a) is

$$\delta\widetilde{I}_q = -\int [\widetilde{\jmath}^\mu\delta\widetilde{\kappa}_\mu + \widetilde{\kappa}_\mu\delta\widetilde{\jmath}^\mu]d^4x \quad (6.6)$$

In the comprehensive action principle $\delta\widetilde{I}_{tot} = 0$, $\widetilde{I}_{tot} = \widetilde{I}_g + \widetilde{I}_m + \widetilde{I}_{em} + \widetilde{I}_q$ we obtain from the coefficients of $\delta\widetilde{\kappa}_\mu$ the variational equation

$$\widetilde{\jmath}^\mu = (4\pi)^{-1}(\phi^2\widetilde{F}^{\mu\nu})_{;\nu}\sqrt{(-g)} = (4\pi)^{-1}[\phi^2\widetilde{F}^{\mu\nu}\sqrt{(-g)}]_{,\nu} \quad (6.7)$$

Hence the identity

$$\widetilde{\jmath}^\mu{}_{,\mu} = 0, \text{ so } \int \widetilde{\jmath}^4 dx^1dx^2dx^3 \text{ conserves } \ (6.8)$$

which reduces by (6.4d) to (6.2) for a point charge. In compliance with this conservation law, we take in (6.6) as familiar in (3.11)

$$\delta\widetilde{\jmath}^\mu = (\widetilde{\jmath}^\nu b^\mu - \widetilde{\jmath}^\mu b^\nu)_{,\nu} \quad (6.9)$$

After integrating by parts we obtain for th second part of (6.6)

$$-\int \widetilde{\kappa}_\mu(\widetilde{\jmath}^{\nu} b^\mu - \widetilde{\jmath}^{\mu} b^\nu)_{,\nu} d^4x = \int \widetilde{F}_{\mu\nu}\widetilde{\jmath}^{\nu} b^\mu d^4x \quad (6.10)$$

For the elements of charged matter the equations of motion obtained from the coefficients of b^μ , by a change of notation similar to (3.9)(3.10)

$$\widetilde{\jmath}^{\nu} = \widetilde{\rho}_e u^\nu \surd(-g)\ ,\ \ \widetilde{\rho}_e = G^{1/2+1/n}\rho_e \quad (6.11a,b)$$

contains a Lorentz force term

$$\widetilde{\rho}_m u^\mu u_{\nu;\mu} + \widetilde{F}_{\mu\nu}\widetilde{\rho}_e u^\nu = 0 \quad (6.12)$$

Dirac, in his book, uses σ instead of our ρ_e for the charge density. The variational equations for $g_{\mu\nu}$ and ϕ are

$$N^\nu_\mu = -8\pi\widetilde{T}^\nu_\mu - 8\pi\widetilde{E}^\nu_\mu,\ \Phi = -8\pi\Phi_{em} \ (6.13a,b)$$

The effect of the factor ϕ^2 in (6.3a) and (6.7) is familiar from Maxwell's theory; as seen from (6.7) in the case of Galelian metric ϕ^2 plays the role of relative dielectric constant ε_ϕ and $(\phi^2)^{-1}$ plays the role of relative magnetic susceptibility μ_ϕ. Hence $\varepsilon_\phi\mu_\phi = 1$ which is consistent with our assumption that c does not evolve.

VII. GEOMETRIC OPTICS

We shall study in this section the propagation of light or electromagnetic wave in space time where the metric and ϕ are given. As is well known in Einstein's theory of general relativity that light moves along a null geodesic, and this can be regarded as a good approximation of geometric optics to wave optics for the electromagnetic equations in the Riemanian space-time. In this section we shall show that this is also true for the theory with varying G, following closely the smilar treatment[5] of the present author for constant G.

From (6.3b) where the covariant derivative can be replaced by ordinary partial derivatives, we can eliminate the cosmic combined potentials and obtain

$$\widetilde{F}_{\mu\nu,\lambda} + \widetilde{F}_{\nu\lambda,\mu} + \widetilde{F}_{\lambda\mu\nu} = 0 \quad (7.1)$$

From (6.7) where charge-current is absent we have

$$[\phi^2\widetilde{F}^{\mu\nu}\surd(-g)]_{,\nu} = [\phi^2 g^{\mu\alpha}g^{\nu\beta}\widetilde{F}_{\alpha\beta}\surd(-g)]_{,\nu} = 0 \quad (7.2)$$

Among the four equations in the set (7.2) there is one differential identity namely the ordinary divergence of its left hand side vanishes identically. The same is true for the four equations in the set (7.1) which can be written in a form similar to (7.2) by introducing the dual of $\widetilde{F}_{\mu\nu}$ defined by

$$\widetilde{\widetilde{F}}^{41} = \widetilde{F}_{23} \; ; \; \widetilde{\widetilde{F}}^{23} = \widetilde{F}_{41} \; ; \; 1, 2, 3 \text{ cyclic} \quad (7.3)$$

Hence the set (7.1), (7.2) contains six independent equations linear in the six dependent variables $\widetilde{F}_{\mu\nu}$. For light or electromagnetic waves, we need to consider solutions of the form of waves

$$\widetilde{F}_{\mu\nu} = \widetilde{f}_{\mu\nu} \sin S \quad (7.4)$$

where the phase S varies quickly in space or time at the scale of a wavelength or a period of light, but the amplitudes $\widetilde{f}_{\mu\nu}$ vary little at such scale, because the multipliers in (7.2) only vary at a scale much much larger. In contrast to the fast variable S we call $\widetilde{f}_{\mu\nu}$, $\phi, g_{\rho\sigma}$ all slow variables. Neglecting the derivatives of the slow variables in comparison with those of the fast variable, we obtain from (7.1) and (7.2) after removing the common factor $\cos S$ and writing $S_{,\lambda} = s_\lambda$ the equations

$$\widetilde{f}_{\mu\nu} s_\lambda + \widetilde{f}_{\nu\lambda} s_\mu + \widetilde{f}_{\lambda\mu} s_\nu = 0 \quad (7.5)$$

$$[\phi^2 g^{\mu\alpha} g^{\nu\beta} \widetilde{f}_{\alpha\beta} \sqrt{(-g)}] s_\nu = 0 \text{ i.e. } s^\beta \widetilde{f}_{\lambda\beta} = 0 \quad (7.6)$$

where $s^\beta = g^{\nu\beta} s_\nu$. In (7.6) the second equation follows from the first by contracting the latter with $g_{\lambda\mu}$ and removing the common factors ϕ^2 and $\sqrt{(-g)}$. These algebraic equations being homogeneous and linear in the amplitudes $\widetilde{f}_{\mu\nu}$, the condition for non-trivial solution is the vanishing of the determinant $s_\lambda s^\lambda = 0$. More simply this can be obtained by contracting (7.5) with s^λ and using (7.6) to obtain $\widetilde{f}_{\mu\nu} s_\lambda s^\lambda = 0$. So at any point where there is light we must have

$$s_\lambda s^\lambda = g^{\lambda\mu} s_\lambda s_\mu = 0 \quad (7.7)$$

Let a real displacement along the light path be denoted by dx^μ, alomg the light path (7.7) always hold, so

$$d(g^{\lambda\mu} s_\lambda s_\mu) = s_\lambda s_\mu g^{\lambda\mu}{}_{,\nu} \, dx^\nu + 2 s^\nu d s_\nu = 0 \quad (7.8)$$

Let a virtual displacement on the surface of constant phase i.e. $S = const.$ be denoted by δx^ν. So we have

$$0 = \delta S = S_{,\nu} \, \delta x^\nu = s_\nu \delta x^\nu \quad (7.9)$$

Since these two displacement vectors are orthogonal to each other, we have also

$$g_{\mu\nu}dx^{\mu}\delta x^{\nu} = 0 \quad (7.10)$$

On comparing the last two equations we see that the coefficients of δx^{ν} there must be proportional. As is well known, we can choose the parameter p for the path of light $x^{\mu} = x^{\mu}(p)$ such that (7.10) agrees with (7.9) and (7.8) is satisfied by the following two set of first order differential equations for the light paths or rays.

$$g_{\mu\nu}(dx^{\mu}/dp) = s_{\nu}, \; ds_{\nu}/dp = -(1/2)g^{\lambda\mu}{}_{,\nu}\, s_{\lambda}s_{\mu} \quad (7.11\text{a,b})$$

In Riemanian geometry, (7.11a,b) is the standard form for the differential equations of null geodesics, the two sets of fiest order equations being completely equivalent to the following one set of second order equations by eliminating s_{ν} with the help of (7.11a),

$$d^2x^{\mu}/dp^2 + \Gamma^{\mu}_{\rho\sigma}(dx^{\rho}/dp)(dx^{\sigma}/dp) = 0 \quad (7.12)$$

and the equation (7.7) being equivalent to the null condition

$$0 = g^{\mu\nu}s_{\mu}s_{\nu} = g^{\mu\nu}g_{\mu\rho}g_{\nu\sigma}(dx^{\rho}/dp)(dx^{\sigma}/dp) = ds^2/dp^2 \quad (7.13)$$

As we have shown that light path in our theory as in Einstein's theory is the null geodesic and the metric outside the sun to the non-cosmological approximation is the same as that in Einstein's theory, so the crucial test about the deflection of light by the sun is also passed by our theory as good as by Einstein's theory.

VIII. QUANTUM MECHANICS

The fundamental natural constant in quantum mechanics is the Planck constant $\hbar$. In order to decide how it evolves, we need some action integral that contains it. It seems most reliable to consider the action integral for Dirac's relativistic theory of electron, because the generalization of Dirac's equation to curved space-time i.e. to general relativity has been considered by many early authors. Here we follow Schrodinger's treatment as summarized in my earlier paper [5] with minor changes in the notation (the electromagnetic potentials A_{μ} there is here written as κ_{μ} and the companion ϕ there of the wave function ψ is here

written as χ). We consider hydrogen like atoms for simplicity. With constant G we have

$$I_D = \int\{(1/2)\chi G^\mu[(i\hbar\partial/\partial x^\mu + e\kappa_\mu)\psi] + (1/2)[(-i\hbar\partial/\partial x^\mu + e\kappa_\mu)\chi]G^\mu\psi - \chi m\psi\}\sqrt{(-g)}d^4x \quad (8.1)$$

where the four general gamma 4×4 matrices G^μ depend only on $g_{\mu\nu}$ satisfying

$$G^\mu G^\nu + G^\nu G^\mu = 2g^{\mu\nu} \quad (8.2)$$

and χ and ψ are respectively 1×4 and 4×1 matrices so that the chain product $\chi G^\mu\psi$ is a number. Here I_D for the hydrogen atom includes the part previously denoted by I_q . We have, cf (6.4a), for Dirac's electron

$$\hat{\jmath}^\mu = -e\chi G^\mu\psi\sqrt{(-g)} \quad (8.3)$$

the ordinary divergence of which vanishes as a consequence of the variational equations with respect to χ and ψ, i.e. the Dirac equation $\delta I_D/\delta\chi = 0$ and its companion $\delta I_D/\delta\psi = 0$ in curved space-time.

$$0 = \chi\frac{\delta I_D}{\delta\chi} - \frac{\delta I_D}{\delta\psi}\psi = [\chi G^\mu\psi\sqrt{(-g)}]_{,\mu} \quad (8.4)$$

The variation of I_D with respect to $g_{\alpha\beta}$ can be obtained from the dependence of G^μ on $g_{\alpha\beta}$ by using the formula derived from (8.2)

$$\partial G^\mu/\partial g_{\alpha\beta} = -(1/4)(G^\alpha g^{\beta\mu} + G^\beta g^{\alpha\mu}) \quad (8.5)$$

According to our rule the modified action integral for varying G is

$$\widetilde{I}_D = \int\{(1/2)\chi G^\mu[(i\widetilde{\hbar}\partial/\partial x^\mu + \widetilde{e}\widetilde{\kappa}_\mu)\psi] + (1/2)[(-i\widetilde{\hbar}\partial/\partial x^\mu + \widetilde{e}\widetilde{\kappa}_\mu)\chi]G^\mu\psi - \chi\widetilde{m}_e\psi\}\sqrt{(-g)}d^4x \quad (8.6)$$

This confirms our previous relations (6.1) for $\widetilde{m}_e$ and (6.2) (6.4c) for $\widetilde{e}\,\widetilde{\kappa}_\mu$. It shows further that the cosmic combined Planck constant

$$\widetilde{\hbar} = G^{1+1/n}\hbar \quad (8.7)$$

conserves in the long.

Meanwhile we call attention to the expression of cosmic combined Coulomb potential $\widetilde{\kappa}_4$ due to a point charge, say $Z\widetilde{e}$ which, in the case of Galelian metric, is not $Z\widetilde{e}/r$ but is

$\widetilde{\kappa}_4 = (\phi^2)^{-1}(Z\widetilde{e}/r) = Z\widetilde{e}/(\varepsilon_\phi r)$ by solving (6.7) with $\mu = 4$. Using (6.2) and (3.1), we verify that the Coulomb potential, only correctly written this way, evolves like (6.4c)

$$\begin{aligned} \widetilde{\kappa}_4 &= G^{1/2}\kappa_4 = G^{1/2}Ze/r \\ \text{so } \widetilde{e}\widetilde{\kappa}_4 &= G^{1+1/n}e\kappa_4 = G^{1+1/n}[Ze^2/r] \quad (8.8) \end{aligned}$$

Hence the fine structure constant defined by (here c being restored)

$$\widetilde{\alpha} = \widetilde{e}^2/(\varepsilon_\phi \widetilde{\hbar} c) = e^2/(\hbar c) = \alpha \quad (8.9)$$

remains unchanged like c during the evolution of the universe.

We are interested in the atomic spectra (of hydrogen say) for light coming from extra-galactic nebulae or from the sun. To obtain the energy levels by quantum mechanics we shall follow closely the treatment in my earlier paper [5] in Einstein's theory. We use the local Galelian metric obtained by a coordinate transformation, which can always be done if we treat the metric $g_{\mu\nu}$ as constants, assuming their value at the position of the atom at the time of emission or absorption of the light quantum. For simplicity consider the diagonal case

$$g_{ii} = -(\lambda_i)^2, i = 1,2,3,\ g_{44} = +(\lambda_4)^2,\ g_{\mu\nu} = 0\ (\mu \neq \nu) \quad (8.10)$$

Then we have

$$G^i = (\lambda_i)^{-1}\gamma^i, i = 1,2,3,\ G^4 = (\lambda_4)^{-1}\gamma^4 \quad (8.11)$$

where γ^μ are the 4×4 gamma matrices familiar for special relativity, defined by

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \quad (8.12)$$

Then Dirac's equation as obtained from (8.6) by varying χ becomes

$$\{\gamma^4(\lambda_4)^{-1}[i\widetilde{\hbar}(\partial/\partial t) + \widetilde{e}\widetilde{\kappa}_4] + \gamma^j(\lambda_j)^{-1}i\widetilde{\hbar}(\partial/\partial x^j) - \widetilde{m}_e\}\psi = 0 \quad (8.13)$$

where the vector potential vanishes and the scalar potential $\widetilde{\kappa}_4$ due to the nucleus at rest at the origin is to be obtained by solving (6.7) with $\mu = 4$, namely

$$[\phi^2 \widetilde{F}^{4\nu}\surd(-g)]_{,\nu} = 4\pi(Z\widetilde{e})\delta(x^1)\delta(x^2)\delta(x^3) \quad (8.14)$$

In the region of atomic scale ϕ may also be treated as constant, so the left hand side of this equation becomes simply $\phi^2 g^{44} g^{jj}\lambda_1\lambda_2\lambda_3\lambda_4(\widetilde{\kappa}_4)_{,jj}$ summed over j. It is convenient to introduce the scaled variables separately

$$y^j = \lambda_j x^j \text{ so } \delta(y^j) = \delta(x^j)/\lambda_j\ j = 1,2,3 \text{ not summed} \quad (8.15)$$

Then (8.14) becomes in the scaled variables

$$\nabla_y^2(\widetilde{\kappa}_4) = -\lambda_4(4\pi Z\widetilde{e}/\phi^2)\delta(y^1)\delta(y^2)\delta(y^3) \quad (8.16)$$

the solution being proportional to the inverse radial distance r_y of the scaled y's

$$\widetilde{\kappa}_4 = \lambda_4(Z\widetilde{e}/\phi^2)r_y^{-1} \quad (8.17)$$

With the help of (8.17), the equation (8.13) can be written in the scaled variables

$$\{\gamma^4[(\lambda_4)^{-1}i\widetilde{\hbar}(\partial/\partial t) + Z\widetilde{e}^2\phi^{-2}r_y^{-1}] + \gamma^j[i\widetilde{\hbar}(\partial/\partial y^j)] - \widetilde{m}\}\psi = 0 \ (8.18)$$

or after removing the common factor $G^{1+1/n}$ (cf.(8.3) (6.1) (6.2) (3.1))

$$\{\gamma^4[(\lambda_4)^{-1}i\hbar(\partial/\partial t) + (Ze^2)r_y^{-1} + \gamma^j[i\hbar(\partial/\partial y^j)] - m\}\psi = 0 \ (8.19)$$

The energy value including the rest energy defined by

$$i\widetilde{\hbar}(\partial/\partial t)\psi = \widetilde{E}\psi \quad (8.20) \quad i\hbar(\partial/\partial t)\psi = E\psi \quad (8.21)$$

is easily obtained by comparing with that of Dirac's equation in his book on quantum mechanics.. Using the quantum numbers n_r and j we obtain (with c restored and using (8.5) for λ_4)

$$\widetilde{E}_{n_r,,j}/\widetilde{m}c^2 = E_{n_r,j}/mc^2 = (g_{44})^{1/2}\{1 + \frac{Z^2\alpha^2}{(n_r + \sqrt{j^2 - Z^2\alpha^2})^2}\}^{-1/2} \ (8.22)$$

For a transition between two quantum states A and B with quantum numbers $n_r(A), j(A)$ and $n_r(B), j(B)$ the frequency $\nu_{A\to B}$ of the atomic spectral line is

$$\nu_{A\to B} = (\widetilde{E}_A - \widetilde{E}_B)/\widetilde{h} = (E_A - E_B)/h = (g_{44})^{1/2}(\nu_{A\to B})_{QM} \ (8.23)$$

where the ordinary quantum mechanical value of the frequency is given by

$$(\nu_{A\to B})_{QM} = \frac{c}{2\pi \lambda\!\!\!^{-}_c}\{1 + \frac{Z^2\alpha^2}{(n_r + \sqrt{j^2 - Z^2\alpha^2})^2}\}^{-1/2} \ \Big|_B^A \ (8.24)$$

Since both the Compton wave length $\lambda\!\!\!^{-}_c = \hbar/mc = \widetilde{\hbar}/\widetilde{m}c = \widetilde{\lambda\!\!\!^{-}}_c$ and the fine structure constant $\alpha = \widetilde{\alpha}$ remain unchanged, we see from (8.24) that the frequency of the atomic spectral line does not change during the evolution. Also the Rydberg constant $R_\infty = mc^2\alpha^2/(4\pi\hbar) = \widetilde{R}_\infty$ remains unchanged. So does the mass ratio e.g. $m_p/m_e = \widetilde{m}_p/\widetilde{m}_e$.

The relation (8.23), it seems, can be generalized to all energy levels of complex atoms and molecules.

We have shown in (8.22) that the gravitational effect on the frequency of the spectral line during emission or absorption in our theory is given by the factor $(g_{44})^{1/2}$, which is the same in our theory as in Einstein's theory to the non-cosmological approximation. So the crutial test about the gravitational red shift of the sun is also passed by our theory as by Einstein's theory.

We note in the Robertson-Walker metric using cosmic time with $g_{44} = 1$ the frequency of the spectral lines emitted anywhere at any time is by (8.23) always the same.

IX. HUBBLE RELATION

In this section we shall investigate the relation between the cosmic red shift of the spectral line received from a cosmic distant source and the distance of that source from an observer here at present. We use the Robertson-Walker metric (5.2), so the frequency of the spectral lines for the distant and nearby object are the same by (8.23), namely ν_{QM} just when they are emitted. But there is a change of the frequency of the spectral line of the distant source during its propagating towards the receiver according to (7.11b).

We denote quantities referring to the distant source by the suffix s and those referring to the receiver or observer here at present by the suffix 0. For simplicity we use geometric optics and radial rays, putting $d\theta = d\varphi = s_\theta = s_\varphi = 0$, the phase of the spherical waves being, as usual, $S = 2\pi(\nu t - r/\lambda) = 2\pi\nu(t - r/\beta)$, with $s_4 = 2\pi\nu, s_r = -2\pi/\lambda$. The null condition gives the velocity of light β for the metric (5.2)

$$\lambda\nu = \beta = R(t_0)/R(t) \quad (9.1)$$

So at the time t_s the wavelength of the spectral line emitted by the source is

$$\lambda_s\nu_{QM} = R(t_0)/R(t_s) \quad (t_s < t_0) \quad (9.2)$$

For the metric (5.2) , being homogeneous, the wave length of this spectra line keeps constant by (7.11b) during the propagation, till the line being recieved and compared at time t_0 with the same line emitted near by with

$$\lambda_0\nu_{QM} = 1 \quad (9.3)$$

The cosmic redshift z is defined by $z = (\lambda_s - \lambda_0)/\lambda_0$, so dividing (9.2) by (9.3) we have

$$1 + z = \lambda_s/\lambda_0 = R(t_0)/R(t_s) \quad (9.4)$$

The optical distance Δ the light propagated is obtained by integrating dr from t_s to t_0

$$\Delta = \int_s^0 dr = \int_{t_s}^{t_0} \frac{R(t_0)}{R(t)} dt \quad (9.5)$$

If this is much larger compared to the linear dimension of the source the inverse square law holds for the intensity and can be used to measure the distance.

For matter dominated cosmological model we have $R(t_0)/R(t) = t_0/t$, so from (9.5) and (9.4) we obtain

$$\Delta = t_0 \log(t_0/t_s) = t_0 \log(1 + z) \quad (9.6)$$

This becomes the Hubble relation

$$z = H_0 \Delta \text{ with } H_0 = (t_0)^{-1} \text{ for } z << 1 \quad (9.7)$$

but for larger z we have from (9.6)

$$z = \exp(H_0 \Delta) - 1 \quad (9.8)$$

X. ELECTROMAGNETIC RADIATION

In the next section we shall supplement the matter dominated universe with electromagnetic radiation. Owing to the appearance of the factor ϕ^2 in the action (6.3a) we need study the electromagnetic waves a bit more in order to obtain the expressions corresponding to (6.5b) and (6.5c) for electromagnetic radiation.

We write at first the electromagnetic wave equations in a form familiar from Maxwell's theory. Free from source (6.7) splits into

$$\begin{aligned} div \overrightarrow{D} &= 0, \quad (10.1) \ \overrightarrow{D} = [\phi^2 \sqrt{(-g)}](\widetilde{F}^{41}, \widetilde{F}^{42}, \widetilde{F}^{43}) \quad (10.1a) \\ curl \overrightarrow{H} - \partial D/\partial t &= 0 \quad (10.2) \ \overrightarrow{H} = [\phi^2 \sqrt{(-g)}](\widetilde{F}^{23}, \widetilde{F}^{31}, \widetilde{F}^{12}) \quad (10.2a) \end{aligned}$$

The other pair of Maxwell's equations come from (6.3b), from which we obtain $\widetilde{F}_{\mu\nu,\lambda} + \widetilde{F}_{\lambda\mu,\nu} + \widetilde{F}_{\nu\lambda,\mu} = 0$ which splits into

$$\begin{aligned} div \overrightarrow{B} &= 0 \quad (10.3) \ \overrightarrow{B} = (\widetilde{F}_{23}, \widetilde{F}_{31}, \widetilde{F}_{12}) \quad (10.3a) \\ curl \overrightarrow{E} + \partial \overrightarrow{B}/\partial t &= 0 \quad (10.4) \ \overrightarrow{E} = (\widetilde{F}_{14}, \widetilde{F}_{24}, \widetilde{F}_{34}) \quad (10.4a) \end{aligned}$$

Comparing (6.5c) with (6.3a), the integrand of the latter being equal to two times $(\overrightarrow{B}\cdot\overrightarrow{H} - \overrightarrow{E}\cdot\overrightarrow{D})$, we obtain

$$\Phi_{em} = -(\overrightarrow{B}\cdot\overrightarrow{H} - \overrightarrow{E}\cdot\overrightarrow{D})/[4\pi\phi\surd(-g)] \quad (10.5)$$

For the Robertson-Walker metric with $k = 0$, it is convenient to use Cartesian coordinates to write

$$ds^2 = dt^2 - f^2(dx^2 + dy^2 + dz^2) \text{ with } f = R(t)/R(t_0) \quad (10.6)$$

Then we have from (10.1a) to (10.4a) the slowly varying permittences ε and μ

$$\begin{aligned} \overrightarrow{D} &= \varepsilon\overrightarrow{E}, \quad (\varepsilon = [\phi^2 f^3]f^{-2}) \quad (10.7a) \\ \overrightarrow{H} &= \mu^{-1}\overrightarrow{B}, \ (\mu^{-1} = [\phi^2 f^3]f^{-4}) \quad (10.7b) \end{aligned}$$

We treat the electromagnetic radiation as superposition of plane electromagnetic waves. For a plane electromagnetic wave propagating along the direction denoted by the unit vector $\overrightarrow{s}$, the only fast variable is the phase $S = \overrightarrow{r}\cdot\overrightarrow{s} - \beta t$ where $\beta = 1/\sqrt{\varepsilon\mu}$, like ε and μ, vary slowly in time and will be treated as constants. Then (we follow here Born and Wolf's treatment[6]) (10.2) and (10.4) becomes

$$\overrightarrow{E} = -\sqrt{\mu/\varepsilon}(\overrightarrow{s}\times\overrightarrow{H}), \overrightarrow{H} = \sqrt{\varepsilon/\mu}(\overrightarrow{s}\times\overrightarrow{E}) \ (10.8)$$

Hence $\overrightarrow{E}$, $\overrightarrow{H}$,and $\overrightarrow{s}$ form a right-handed orthogonal triad, and we have

$$\varepsilon E^2 = \mu H^2 \text{ or } \overrightarrow{D}\cdot\overrightarrow{E} - \overrightarrow{H}\cdot\overrightarrow{B} = 0 \quad (10.9)$$

So by (10.5) we have $\Phi_{em} = 0$ for a plane electromagnetic wave, and by superposition of plane waves we obtain for electromagnetic radiation

$$\Phi_{em} = 0 \text{ (electromagnetic radiation)} \quad (10.10)$$

For electromagnetic field where charge-current density vanishes we have the differential identity,

$$\widetilde{E}^{\nu}_{\mu;\nu} + \Phi_{em}\phi_{;\mu} = 0 \quad (10.11)$$

which can easily be verified from (6.5b) and (6.5c). Where charge-current density does not vanish, this identity should be replaced by

$$\widetilde{T}^{\nu}_{\mu;\nu} + \widetilde{E}^{\nu}_{\mu;\nu} + \Phi_{em}\phi_{;\mu} = 0 \quad (10.12)$$

so that the terms involving Lorentz's force from $\widetilde{T}^{\nu}_{\mu;\nu}$ and $\widetilde{E}^{\nu}_{\mu;\nu}$ cancel. For electromagnetic radiation (10,11) simplifies by (10.10) to

$$\widetilde{E}^{\nu}_{\mu;\nu} = 0 \text{ (electromagnetic radiation)} \quad (10.13)$$

For homogeneous and isotropic electromagnetic radiation the superposition of plane waves propagating in all directions leaves only the diagonal elements of $\widetilde{E}^{\nu}_{\mu}$ non-vanishing Since from (6.5b) we have zero trace before the superposition, we conclude that

$$\widetilde{E}^1_1 = \widetilde{E}^2_2 = \widetilde{E}^3_3 = -\widetilde{p}_r = -\widetilde{\rho}_r/3, \; \widetilde{E}^4_4 = \widetilde{\rho}_r \; (10.14)$$

It is well known that with (10.14) and the metric (10.6) that (10.13) can be integrated to obtain

$$\widetilde{\rho}_r = \widetilde{\rho}_r(t) = \widetilde{\rho}_r(t_0)R^4(t_0)/R^4 \quad (10.15)$$

XI. MATTER PLUS ELECTROMAGNETIC RADIATION

We now consider homogeneous cosmology including matter and electromagnetic radiation, $\widetilde{I}_{tot} = \widetilde{I}_g + \widetilde{I}_m + \widetilde{I}_{em}$ and write according to our theory with varying G the equations for the comprehensive variational principle $\delta\widetilde{I}_{tot} = 0$.

The variational equation for ϕ remains unchanged as (3.17) because of (10.10). The variational equations for $g_{\mu\nu}$ are

$$N^4_4 = -8\pi(\widetilde{\rho}_m + \widetilde{\rho}_r), \quad N^1_1 = 8\pi\widetilde{p}_r = 8\pi\widetilde{\rho}_r/3 \quad (11.1)$$

We give these equations in full, cf. (4.8), (4.9) and (4.10)

$$\begin{aligned} \Phi &= 6[\ddot{R}/R + \dot{R}^2/R^2]\phi \\ + 8[\ddot{\phi} + 3(\dot{R}/R)\dot{\phi}] &= 0 \qquad\qquad (11.2) \end{aligned}$$

$$\begin{aligned} N^1_1 &= \phi^2[-\dot{R}^2/R^2 - 2\ddot{R}/R] - (\phi^2)^{\cdot\cdot} - 2(\phi^2)^{\cdot}\dot{R}/R \\ +4\dot{\phi}^2 &= (8\pi/3)\widetilde{\rho}_r(t_0)R^4(t_0)/R^4 \qquad\qquad (11.3) \end{aligned}$$

$$\begin{aligned} N^4_4 &= \phi^2[-3\dot{R}^2/R^2] - 3(\phi^2)^{\cdot}\dot{R}/R - 4\dot{\phi}^2 \\ &= -8\pi[\widetilde{\rho}_m(t_0)R^3(t_0)/R^3 + \widetilde{\rho}_r(t_0)R^4(t_0)/R^4] \qquad (11.4) \end{aligned}$$

The differential identity, cf. (3.6), $N^{\beta}_{4;\beta} + \Phi\phi_{;4} = 0$ shows that among the three equations (11.2,3,4) only two are independent, the condition for compatiability on the right hand side being separately guaranteed for matter and for electromagnetic radiation. The combination $N^{\nu}_{\nu} + \Phi\phi$ cf (3.18) remains unchanged and can be integrated to give, cf. (4.2),

$$(\phi^2)^{\cdot} R^3/R^3(t_0) = -8\pi\widetilde{\rho}_m(t_0)t \quad (11.5)$$

which can be used in place of any equation among (11.2,3,4). We may take as the two independent equations (11.4) and (11.5), and use the new variables

$$\tau = t/t_0, \xi = R/R(t_0), \eta = \phi^2/\phi^2(t_0) \ (11.6)$$

and parameters

$$\begin{aligned} 8\pi t_0^2\widetilde{\rho}_m(t_0)/\phi^2(t_0) &= 8\pi t_0^2 G(t_0)\rho_m(t_0) = A \qquad (11.7) \\ \widetilde{\rho}_r(t_0)/\widetilde{\rho}_m(t_0) &= \rho_r(t_0)/\rho_m(t_0) = \varepsilon_{r/m} \sim (10^{-2} \text{ to } 10^{-4}) \quad (11.8) \end{aligned}$$

Then (11.4) and (11.5) become, with circle denoting $d/d\tau$,

$$-3\eta\mathring{\xi}^2/\xi^2 - 3\mathring{\eta}\mathring{\xi}/\xi - \mathring{\eta}^2/\eta = -A/\xi^3 - \varepsilon_{r/m}A/\xi^4 \ (11.9)$$

$$\mathring{\eta}\xi^3 = -A\tau \quad (11.10)$$

The solution for matter dominated universe given in §4 is for $\varepsilon_{r/m} = 0$ that $\xi = \tau$, $\eta = 1/\tau$, and $A = 1$. This solution satisfies not only the "initial" conditions at $\tau = 1$ that $\xi = 1$ and $\eta = 1$ but also the "final" condition at $\tau = 0$ that $\xi = 0$. The extra condition serves to determine the parameter A. We shall leave the mathematical problem of solving these equations (11.9), (11.10) but only note that the asymptotic solution of these equations for large ξ is that of matter dominated universe. Only with the help of the asymptotic solution which is simple enough to reveal the cosmical combined conservation laws that we find the way to modify or to generalize Einstein's theory of general relativity to work to cosmological time. For small values of τ, at $\tau \sim \varepsilon_{r/m}$, the deviation from the asymptotic solution will be appreciable. For smaller values of τ refinement of the action integral for matter with corresponding refinement of $\widetilde{T}_{\mu\nu}$ need to be considered to take account of the internal energy and pressure, which at thermal equilibrium can be treated according to the principle of statistical mechanics.

XII. STATISTICAL MECHANICS

The fundamental natural constant in statistical mechanics is the Boltzmann constant k_B, in this section we shall see how it evolves in cosmological long time. For this purpose we need to consider for example black body radiation where k_B appears in Planck's formula and in the energy density of black body radiation.

$$\rho_r = \int_0^\infty \frac{8\pi h\nu^3/c^3}{\exp(h\nu/k_B T) - 1} d\nu = \frac{\pi^2}{15c^3}\frac{k_B^4}{\hbar^3}T^4 \quad (12.1)$$

Comparing this with the cosmic combined expression

$$\widetilde{\rho}_r = \int_0^\infty \frac{8\pi \widetilde{h}\nu^3/c^3}{\exp(\widetilde{h}\nu/\widetilde{k}_B T) - 1} d\nu = \frac{\pi^2}{15c^3}\frac{(\widetilde{k}_B)^4}{(\widetilde{\hbar})^3}T^4 \quad (12.2)$$

and noting that like ρ_m (cf. (11.1) and (2.5)) we must have $\widetilde{\rho}_r = G^{1+1/n}\rho_r$, as we know that electromagnetic radiation can be regarded as matter, namely photons. From this we obtain from (12.1) (12.2) and (8.7) the similar relation

$$\widetilde{k}_B = G^{1+1/n} k_B \quad (12.3)$$

Thus we have the general rule for c.c. energy, cf. (2.5), (8.7), (12.3), and also cf.(8.8) for the c.c. Coulomb energy,

$$\widetilde{E} = G^{1+1/n} E, \ (\widetilde{E} = \widetilde{m}c^2, \widetilde{h}\nu, \widetilde{k}_B T, \widetilde{e}\widetilde{\kappa}_4) \quad (12.4)$$

where c^2, ν, T do not change. Since energy, momentum and stress form a four dimensional tensor, the above relation (12.4) holds too for the momentum $\widetilde{p}_x, \widetilde{p}_y, \widetilde{p}_z$ and the pressure $\widetilde{P}$, while velocity u_x, u_y, u_z and volume V do not change. This makes the Boltzmann factor

$$\exp(-\widetilde{E}/\widetilde{k}_B T) = \exp(-E/k_B T) \quad (12.5)$$

the same and the counting of phase cells

$$V d\widetilde{p}_x d\widetilde{p}_y d\widetilde{p}_z/(\widetilde{h})^3 = V dp_x dp_y dp_z/h^3 \quad (12.6)$$

also the same, so for canonical ensemble of systems at local thermal equilibrium, the partition function (Zustandsumme) $\widetilde{Z} = Z$ and hence the distribution function is the same with or without the cosmic combination factor. For example, for Bose-Einstein statistics, the

distribution function of photons is given by, cf (12.1) and (12.2), where the factor two accounts for the two polarizations

$$\frac{(2)4\pi\nu^2/c^3}{\exp(\widetilde{h}\nu/\widetilde{k}_B T)-1} = \frac{(2)4\pi\nu^2/c^3}{\exp(h\nu/k_B T)-1} \quad (12.7)$$

We shall not go into the details of the early universe but suffice it to note that at high temperature the internal energy and pressure must be taken into account. In Einstein's theory the action integral I_m (E2.3a) for matter without internal stress is replaced by that of perfect fluid I_{fl}, {we follow here Fock's treatment in his book [7] space time and gravitation, eq.(48.28) et seq.}, the cosmic combined expression in our theory is given accordingly by

$$\begin{aligned}\widetilde{I}_{fl} &= -\int F(\widetilde{\rho}_m)\sqrt{(-g)}d^4x \\ &= -\int \widetilde{\rho}_m(1+\widetilde{\epsilon}_m)\sqrt{(-g)}d^4x \ (12.8)\end{aligned}$$

Here $\widetilde{\epsilon}_m$ denotes the c.c. internal energy per unit c.c. mass $\widetilde{\rho}_m$ of the ideal fluid and is considered as a function of the latter only, as we are dealing with adiabatic processes in which the c.c. entropy per unit c.c. mass, denoted here by $\widetilde{s}_m$ is kept constant, i.e.

$$d\widetilde{\epsilon}_m + \widetilde{P}d(1/\widetilde{\rho}_m) = Td\widetilde{s}_m = 0 \quad (12.9)$$

To find the variation of $\widetilde{I}_{fl}$ we express $\widetilde{\rho}_m$ in terms of $\widetilde{\wp}^\mu$ by (2.4a) and obtain

$$\begin{aligned}\delta\widetilde{I}_{fl} &= -\int \frac{dF}{d\widetilde{\rho}_m}\delta(\widetilde{\rho}_m\sqrt{-g})d^4x \\ &\quad + \int(\widetilde{\rho}_m\frac{dF}{d\widetilde{\rho}_m} - F)(\delta\sqrt{-g})d^4x \\ &= -(1/2)\int(\widetilde{T}^{\mu\nu}_{fl}\delta g_{\mu\nu})\sqrt{(-g)}d^4x \\ &\quad - \int \frac{dF}{d\widetilde{\rho}_m}u_\mu\delta\widetilde{\wp}^\mu d^4x \quad (12.10)\end{aligned}$$

with

$$\begin{aligned}\widetilde{T}^{\mu\nu}_{fl} &= \widetilde{\rho}_m\frac{dF}{d\widetilde{\rho}_m}u^\mu u^\nu - (\widetilde{\rho}_m\frac{dF}{d\widetilde{\rho}_m} - F)g^{\mu\nu} \\ &= [\widetilde{\rho}_m(1+\widetilde{\epsilon}_m)+\widetilde{P}]u^\mu u^\nu - \widetilde{P}g^{\mu\nu} \ (12.11)\end{aligned}$$

Using (3.11) in compliance with the modified constraint (2.3b) we obtain from the second part of (12.10) the following equations of motion for the ideal fluid

$$\begin{aligned}&\widetilde{\rho}_m\frac{dF}{d\widetilde{\rho}_m}u^\nu u_{\mu;\nu} + \widetilde{\rho}_m u^\nu u_\mu(\frac{dF}{d\widetilde{\rho}_m})_{;\nu} - \widetilde{\rho}_m(\frac{dF}{d\widetilde{\rho}_m})_{;\mu} \\ &= [\widetilde{\rho}_m(1+\widetilde{\epsilon}_m)+\widetilde{P}]u^\nu u_{\mu;\nu} + u^\nu u_\mu\widetilde{P}_{;\nu} - \widetilde{P}_{;\mu} = 0 \quad (12.12)\end{aligned}$$

One can verify the covariant divergence identity

$$\widetilde{T}^{\nu}_{fl.\mu;\nu} \;=\; (\widetilde{\rho}_m u^{\nu})_{;\nu}\frac{dF}{d\widetilde{\rho}_m}u_{\mu} + \widetilde{\rho}_m u^{\nu}(\frac{dF}{d\widetilde{\rho}_m}u_{\mu})_{;\nu} - (\widetilde{\rho}_m\frac{dF}{d\widetilde{\rho}_m} - F)_{;\mu} = 0 \tag{12.13}$$

with the help of (12.12) and (2.3b) (2.4b). We note that $\widetilde{I}_{fl}$, like $\widetilde{I}_m$, does not involve ϕ, hence contributes nothing to the variational equation with respect to ϕ.

In the specific internal energy $\widetilde{\epsilon}_m$, being energy per unit mass, according to (12.4), the cosmic combination factor will cancel. Thus we expect $\widetilde{\epsilon}_m = \epsilon_m$. This can be easily verified, e.g. for an ideal gas at low temperature. With Maxwell's law for the distribution of velocity $\exp[-\widetilde{m}(u_x^2+u_y^2+u_z^2)/(2\widetilde{k}_BT)]du_xdu_ydu_z$ in which $\widetilde{m}$ and $\widetilde{k}_B$ can be simultaneously be replced by m and k_B, the translational internal energy is $\widetilde{\rho}_m\widetilde{\epsilon}_m = 3n_m\widetilde{k}_BT/2$ while $\widetilde{\rho}_m = n_m\widetilde{m}$ where $\widetilde{m}$ being the c.c. conservered mass of one molecule, and n_m the number density of the molecules. Similarly $\widetilde{\epsilon}_m = \epsilon_m$ will hold for relativistic gas at high temperatures.

XIII. SUMMARY AND REMARK

From our theory with varying G we have sysmatically found the cosmic combined natural constants $\widetilde{m}, \widetilde{\hbar}, \widetilde{k}_B, \widetilde{e}$

$$\widetilde{m}/m = \widetilde{\hbar}/\hbar = \widetilde{k}_B/k_B = [G/G_0]^{1+1/n}, \widetilde{e}/e = [G/G_0]^{1/2+1/n} \tag{13.1}$$

which remain to be constant in the long. Our present fundamental principles of quantum mechanics and statistical mechanics work in the long with $\widetilde{m}$, $\widetilde{\hbar}$ and $\widetilde{k}_B$. It is due to the old age of the present universe that the combination factors $G^{1+1/n}$ involved in these c.c. constants vary very slowly by now that we take m, $\hbar$ and k_B as natural constants. The law of electromagnetism in the long differs from the law at present by a dielectric and magnetic permittivity $\varepsilon = \mu^{-1} = [G/G_0]^{1/n}$. The velocity of light c, the Compton wave length for a particle $\widetilde{\hbar}/\widetilde{m}c$, the fine structure constant $\widetilde{e}^2/(\varepsilon\widetilde{\hbar}c)$ and the proton-electron mass ratio $\widetilde{m}_p/\widetilde{m}_e$ all remain unchanged during the evolution.

For phenomena occurring outside the gravitating body like the crucial tests our theory to the non-cosmological approximation reduces to Einstein's theory. Difference arises for phenomena occurring inside the gravitating body, including the case of cosmology. E.g. for matter-dominated universe our theory with varying G yields for the expansion law $R \propto t$ between the radius and the age of universe so as to be in agreement with the empirical

large number equality $GM \propto t$, while in Einstein's theory with constant G and conservation of mass we have $R \propto t^{2/3}$ with GM = constant of course. In our theory with varying G , a tensor term automatically arises from the spatial and temporal derivatives of G. This tensor automatically vanishes in a region where G is constant, as is the case with the exterior solution of the sun when we consider the crucial tests of general relativity. In the problem of non-static homogeneous cosmology this tensor with non-vanishing components $\Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = 1/t^2$, $\Lambda_4^4 = 2/t^2$ (Or less probably $\Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = 2/t^2$, $\Lambda_4^4 = 1/t^2$) takes the place of cosmological constant. It seems desirable to make an analysis of the observational data from the beginning according to the present theory with varying G. It would be also interesting to apply the theory with varying G to the interior solution of some high density objects where spatial variation of G is appreciable.

XIV. ACKNOWLEDGMENTS

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